$$\begin{cases} u_{tt} = c^2 u_{xx}, & 0<x<+\infty,\ t>0 \\ u(x,0) = f(x),\ u_t(x,0) = g(x) & 0<x<+\infty, \\ u_x(0,t) + h u(0,t) = 0, & t\geq 0 \end{cases}$$

Sol^n.: For $x>ct$, the solution is given by

$$u(x,t) = \phi(x+ct) + \psi(x-ct) = \frac{1}{2}\left[f(x+ct)+f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

For $x<ct$, we need to make use of

$$\phi'(ct) + \psi'(-ct) + h[\phi(ct)+\psi(-ct)] = 0 \text{ or } \psi'(t) + h\psi(t) = -(\phi'(-t) + h\phi(-t)).$$

The solution of $\psi(t)$ is given by

$$\psi(t) = C(t)e^{-ht}, \text{ where } C(t) = \phi(0) - e^{ht}\phi(-t) + 2\int_0^{-t} e^{-hs}\phi'(s)\,ds + C$$

$$\Rightarrow \psi(t) = -\phi(-t) + e^{-ht}[\phi(0)+C] + 2e^{-ht}\int_0^{-t} e^{-hs}\phi'(s)\,ds.$$

When $t=0 \Rightarrow C = \psi(0) = \frac{1}{2}f(0) - \frac{k}{2} \Rightarrow \psi(t) = -\phi(-t) + f(0)e^{-ht} + 2e^{-ht}\int_0^{-t} e^{-hs}\phi'(s)\,ds.$

Since $\phi(\xi) = \frac{1}{2}f(\xi) + \frac{1}{2c}\int_0^{\xi} g(s)\,ds \Rightarrow 2e^{-ht}\int_0^{-t} e^{-hs}\left(\frac{1}{2}f'(s) + \frac{1}{2c}g(s)\right)ds$

$$= e^{-ht}\left[e^{-hs}f(s)\right]_0^{-t} + he^{-ht}\int_0^{-t} e^{-hs}f(s)\,ds = \cancel{e^{-2ht}}\,f(-t) - f(0)e^{-ht} + he^{-ht}\int_0^{-t} e^{-hs}f(s)\,ds + \cdots$$

$+ \cdots$ ↳ g-term.

$$\Rightarrow \psi(t) = -\phi(-t) + \cancel{e^{-2ht}}\,f(-t) + he^{-ht}\int_0^{-t} e^{-hs}f(s)\,ds + e^{-ht}\int_0^{-t} e^{-hs}\cdot\frac{1}{c}g(s)\,ds$$

$$\Rightarrow \psi(x-ct) = -\phi(ct-x) + f(ct-x) + he^{-ht}\int_0^{ct-x} e^{-hs}f(s)\,ds + \frac{1}{c}e^{h(ct-x)}\int_0^{ct-x} e^{-hs}g(s)\,ds.$$

$$= \frac{1}{2}f(ct-x) - \frac{1}{2c}\int_0^{ct-x} g(s)\,ds - \frac{k}{2} + he^{-ht}\int_0^{ct-x} e^{-hs}f(s)\,ds + \frac{1}{c}e^{h(ct-x)}\int_0^{ct-x} e^{-hs}g(s)\,ds$$

$$\Rightarrow u(x,t) = \frac{1}{2}\left[f(x+ct)+f(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)\,ds + he^{-ht}\int_0^{ct-x} e^{-hs}f(s)\,ds + \frac{1}{c}e^{h(ct-x)}\int_0^{ct-x} e^{-hs}g(s)\,ds.$$