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Lecture 8.

Determinant point process & Christoffel-Darboux kernel: Revisited

Notes

In this part, we apply the point process & distribution of eigenvalues for unitary invariant ensemble. Firstly, we show that the correlation function could be written as a determinant.

1. Determinant point process (DPP)

Def. 8.1 (DPP):

Let $\mathbb{P}$ be a point process on $\mathbb{R}$ with correlation function ρ_k. The point process is determinantal if there exists a kernel $K(x,y)$, s.t.

$$\rho_k(x_1,\cdots,x_k) = \det\left[K(x_i,x_j)\right]_{i,j=1}^{k}.$$

The kernel K is called the correlation kernel of the DPP.

Remark 8.2: The correlation kernel is not unique. Obviously, $\det\left[K(x_i,x_j)\right]_{i,j=1}^{k} = \det\left[\frac{h(x_j)}{h(x_i)}K(x_i,x_j)\right]_{i,j=1}^{k}$

The following example is to show how to construct a symmetric probability density from a correlation kernel.

Theorem 8.3: Suppose the kernel $K(x,y)$ satisfies:

① $\int_{-\infty}^{+\infty} K(x,x)\,dx = n$;

② For every $x_1,\cdots,x_n \in \mathbb{R}$, $\det\left[K(x_i,x_j)\right]_{i,j=1}^{n}$ is non-negative;

③ K has the reproducing property, i.e. for every $x,z \in \mathbb{R}$,

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$$\int_{\mathbb{R}} K(x,y)K(y,z)\,dy = K(x,z),$$

then $p(x_1,\cdots,x_n) = \frac{1}{n!}\det\left[K(x_i,x_j)\right]_{i,j=1}^{n}$ is a symmetric probability density function on $\mathbb{R}^n$. The associated n-point process is a DPP with K as correlation kernel.

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<u>proof</u>: ① According to the result that

$$\int_{\mathbb{R}^n} p(x_1,\cdots,x_n)\,dx_1\cdots dx_n = 1,$$

and $\det[K(x_i,x_j)]_{i,j=1}^n$ is non-negative for all $x_i\in\mathbb{R}$, it is known $p(x_1,\cdots,x_n) = \frac{1}{n!}\det[K(x_i,x_j)]_{i,j=1}^n$ is a probability density function;

② Integrating-out formula. Here we only show that for $k=n-1$, i.e.

$$\det[K(x_i,x_j)]_{i,j=1}^{n-1} = \int_{\mathbb{R}} \det[K(x_i,x_j)]_{i,j=1}^n\,dx_n.$$

Since

$$\det[K(x_i,x_j)]_{i,j=1}^n = K(x_n,x_n)\det[K(x_i,x_j)]_{i,j=1}^{n-1} + \sum_{l=1}^{n-1}(-1)^{n+l}K(x_n,x_l)\det[K(x_i,x_j)]_{i,j=1,\,i\neq n,\,j\neq l}^n$$

and $\int_{\mathbb{R}} K(x_n,x_n)\,dx_n = n$, we have

$$\int_{\mathbb{R}}\det[K(x_i,x_j)]_{i,j=1}^n\,dx_n = n\cdot\det[K(x_i,x_j)]_{i,j=1}^{n-1} + \sum_{l=1}^{n-1}(-1)^{n+l}\int_{\mathbb{R}} K(x_n,x_l)\det\left[K(x_i,x_j)\ K(x_i,x_n)\right]_{\substack{i=1,\cdots,n-1\\ j=1,\cdots,\hat{l},\cdots,n}} dx_n$$

$$= n\cdot\det[K(x_i,x_j)]_{i,j=1}^{n-1} + \sum_{l=1}^{n-1}(-1)^{n+l}(-1)^{n-1-l}\cdot\det\left(K(x_i,x_j)\right)_{i,j=1}^{n-1}$$

$$= \det[K(x_i,x_j)]_{i,j=1}^{n-1}. \qquad \#$$

2. Christoffel-Darboux kernel as correlation kernel

We first show that the eigenvalue dist. of unitary invariant ensemble is a DPP. In other words, we can write the density function as a determinant.

Summary

<u>Thm 8.4</u>: We have $\frac{1}{\tilde{Z}_n}\prod_{1\le i<j\le n}(\lambda_j-\lambda_i)^2\prod_{i=1}^n e^{-V(\lambda_i)} = \frac{1}{n!}\det[K(\lambda_i,\lambda_j)]_{i,j=1}^n$, where $\tilde{Z}_n = \int_{\mathbb{R}^n}\prod_{1\le i<j\le n}(\lambda_j-\lambda_i)^2\prod_{i=1}^n e^{-V(\lambda_i)}\,d\lambda_1\cdots d\lambda_n$, $K(x,y) = e^{-\frac{1}{2}(V(x)+V(y))}\sum_{j=0}^{n-1}p_j(x)\cdot p_j(y)$, and $\{p_i(x)\}_{i=0}^{+\infty}$ is a family of orthonormal polynomials w.r.t. the weight $e^{-V(x)}$.

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~~proof~~. Recall that if $\{P_j(x)\}_{j=0}^{+\infty}$ is a family of orthonormal polynomials w.r.t the weight $e^{-V(x)}$, then we have $\int_{-\infty}^{+\infty} P_j(x)P_k(x)e^{-V(x)}dx=\delta_{j,k}$, and the coefficient of the highest degree is given by $\gamma_n=\left(\frac{I_n}{I_{n+1}}\right)^{1/2}$, where $I_n=\frac{1}{n!}\tilde{Z}_n$. Therefore, by noting that

$$\prod_{1\le i<j\le n}(\lambda_j-\lambda_i)=\det\left(\lambda_i^{j-1}\right)_{i,j=1}^{n}=\det\left[\mathcal{P}_{k-1}(\lambda_j)\right]_{j,k=1}^{n}$$

$$=\left(\prod_{j=0}^{n-1}\gamma_j\right)^{-1}\det\left[P_{k-1}(\lambda_j)\right]_{j,k=1}^{n},$$

~~and~~ we have

$$\frac{1}{\tilde{Z}_n}\prod_{1\le i<j\le n}(\lambda_j-\lambda_i)^2\prod_{i=1}^{n}e^{-V(\lambda_i)}$$

$$=\frac{1}{n!}\det\left(P_{k-1}(x_j)e^{-\frac{1}{2}V(x_j)}\right)_{j,k=1}^{n}\cdot\det\left(P_{m-1}(x_\ell)e^{-\frac{1}{2}V(x_\ell)}\right)_{m,\ell=1}^{n},$$

and thus the proof is complete. #

As we have shown, the Christoffel-Darboux kernel is a reproducing kernel and $\int_{-\infty}^{+\infty}K_n(x,x)dx=n$, according to the orthogonality. Therefore, the eigenvalue dist. and 1-point correlation function for unitary invariant ensemble could be transformed into the C-D kernel. Explicitly, we have

$$\rho_1(x)=a_n e^{-V(x)}\left(P_n'(x)P_{n-1}(x)-P_n(x)P_{n-1}'(x)\right).$$

Summary

In this case, the gap probability could be written as

$$P_A(0)=1+\sum_{k=1}^{+\infty}\frac{(-1)^k}{k!}\int_{A^k}\det\left[K(x_i,x_j)\right]_{i,j=1}^{k}dx_1\cdots dx_k:=\det(I-K_A),$$

which is called a Fredholm determinant of the (trace class) operator $K_A: L^2(A)\to L^2(A)$, $f\mapsto K_Af$

where $K_Af:=\int_{\cancel{\mathbb{R}}A}K(x,y)f(y)\,dy,\ x\in A.$

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In general, we have $\det(I+zK_A) = 1+\sum_{k=1}^{+\infty}\frac{z^k}{k!}\times \int_{A^k}\det[K(x_i,x_j)]_{i,j=1}^{k}dx_1\cdots dx_k$, which converges for all $z\in\mathbb{C}$, and defines an entire function in the complex plane.

As we said, we are interested in the limiting behavior of the dist. of eigenvalues, which is now transformed into the limiting behaviour of correlation kernel, which is the main topic in the subsequent lectures. Before we move to the limiting behavior, we need to discuss some properties of the correlation kernel.

3. Properties of the correlation kernel

① Limit of the correlation kernel

Prop. 8.5: Suppose that for each n, K_n is the correlation kernel of a DPP $\mathbb{P}_n$, and that K is a kernel s.t.

$$\lim_{n\to\infty} K_n(x,y) = K(x,y)$$

uniformly for x,y in compact subsets of $\mathbb{R}$. Then K is also the correlation kernel of a DPP $\mathbb{P}$, and $\mathbb{P}$ is the weak limit of $\{\mathbb{P}_n\}$.

Summary

However, the limiting behavior is not simply by taking $n\to\infty$, but need to consider some scaling limit. Recall the CLT: If $x_1,\cdots,x_n$ are i.i.d. with $\mathbb{E}[x_i]=\mu$ and $\mathrm{var}[x_i]=\sigma^2$, then $\lim_{n\to\infty}\frac{1}{\sqrt{n}}\cdot\frac{\bar{x}-\mu}{\sigma} = \mathcal{N}(0,1)$. Therefore, we need to consider the limit under scaling and translation.

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② Scalings and translations

Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous bijection. Then

$$\mathcal{X} \mapsto f(\mathcal{X}) = \{f(x) \mid x \in \mathcal{X}\}$$

maps configurations on $\mathbb{R}$ to configurations on $\mathbb{R}$. The push forward mapping of a point process $\mathbb{P}$ is again a point process, denoted by $f(\mathbb{P})$.

<u>prop. 8.6</u>: Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous bijection and $g = f^{-1}$. Assume that g is differentiable. If K is the kernel of a DPP $\mathbb{P}$, then $f(\mathbb{P})$ is also a DPP and

$$\tilde{K}(x,y) = \sqrt{g'(x)g'(y)}\, K(g(x), g(y)).$$

<u>proof</u>: By substituting $x_i = f^{-1}(y_i) := g(y_i)$, we have

$$\rho(x_1, \cdots, x_n)\,dx_1 \cdots dx_n = \rho(f^{-1}(y_1), \cdots, f^{-1}(y_n))\, df^{-1}(y_1) \cdots df^{-1}(y_n)$$

$$= \rho(g(y_1), \cdots, g(y_n))\, g'(y_1) \cdots g'(y_n)\, dy_1 \cdots dy_n$$

$$= \frac{1}{n!} \det\left[K(g(y_i), g(y_j))\sqrt{g'(y_i)g'(y_j)}\right] dy_1 \cdots dy_n.$$

Therefore, if we denote $\tilde{K}(x,y) = \sqrt{g'(x)g'(y)}\, K(g(x), g(y))$, then

$$\rho(x_1, \cdots, x_n)\,dx_1 \cdots dx_n = \tilde{\rho}(y_1, \cdots, y_n)\,dy_1 \cdots dy_n = \frac{1}{n!}\det\left[\tilde{K}(y_i, y_j)\right]_{i,j} dy_1 \cdots dy_n. \ \#$$

Summary

A simple, but important case is when $f(x) = c(x - x^*)$ with $c > 0$ and $x^* \in \mathbb{R}$. Then $\mathbb{P} \to f(\mathbb{P})$ amounts to a centering and scaling of the point process. The corresponding correlation kernel is $\tilde{K}(x,y) = \frac{1}{c} K(x^* + \frac{x}{c}, y^* + \frac{y}{c})$. In the following, we'll discuss the limit of the rescaled kernel $K(x,y) = \lim_{n\to\infty} \frac{1}{cn^\gamma} K_n(x^* + \frac{x}{cn^\gamma}, y^* + \frac{y}{cn^\gamma})$.