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Cues

Lecture 7.

Correlation kernels of random matrices

When we study random matrices, we're interested in:
① the dist. of eigenvalues;
② the largest/smallest eigens;
③ the gap between two eigens;
④ the expectation of eigenvalues located in an exact interval.

We have already transformed the distribution of random matrices into dist. of eigenvalues, which could be viewed as n points on $\mathbb{R}^n$. Therefore, we need to introduce some basic concepts in point process.

Notes

1. Point Process

Def. 7.1 (Point process):

A configuration space $\mathcal{X}$ is a subset of $\mathbb{R}$ which is locally finite in the sense that $\#(\mathcal{X}\cap K)<+\infty$ for every compact subset $K\subseteq\mathbb{R}$. A point process on $\mathbb{R}$ is a probability measure on the space of all configurations.

Remark: The definition of configuration could be extended from the real line to other locally compact topological spaces.

Def. 7.2 (n-point process):

A point process $\mathbb{P}$ is an n-point process if $\mathbb{P}[\#\mathcal{X}=n]=1$.

Example: Given a probability measure μ on $\mathbb{R}$ and a positive integer n, we create a point process, by picking n independent random points according to μ. If $\mu(\{a\})=0$ for every $a\in\mathbb{R}$, then the points will be distinct a.s.

A probability density function $p(x_1,\cdots,x_n)$ on $\mathbb{R}^n$ is symmetric if it is invariant under permutation of coordinates, i.e. $p(x_{\sigma(1)},\cdots,x_{\sigma(n)})=p(x_1,\cdots,x_n)$, $\forall\sigma\in S_n$.

Summary

Prop. 7.3. A symmetric probability density function $p(x_1,\cdots,x_n)$ defines an n-point process.

proof: Obviously, if $(x_1,\cdots,x_n)\in\mathbb{R}^n$ is randomly taken according to p, then $x_i\neq x_j$ for $i\neq j$ a.s., and $\mathcal{X}=\{x_1,\cdots,x_n\}$ is a random subset of $\mathbb{R}$ satisfying $\mathbb{P}(\#\mathcal{X}=n)=1$. #

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Therefore, the distribution of eigenvalues for unitary invariant ensemble induces an n-point process on $\mathbb{R}$.

2. Correlation function

Let $\mathbb{P}$ be a point process on $\mathbb{R}$. Expectation values $\mathbb{E}$ are taken with respect to this point process

$$\mathbb{E}: A \mapsto \mathbb{E}[\#(\chi \cap A)], \qquad (*)$$

where A is the Borel sets in $\mathbb{R}$.

Def. 7.4 (intensity / 1-point correlation function): Suppose that (*) is absolutely continuous with respect to Lebesgue measure. Then its density ρ_1 is called the intensity or 1-point correlation function of the point process.

If ρ_1 is the intensity, then

$$\mathbb{E}[\#(\chi \cap A)] = \int_A \rho_1(x)\,dx, \qquad (**)$$

and $\rho_1(x)dx$ is the probability to have a point in the infinitesimal interval $[x, x+dx]$. Especially, if we consider $\mathbb{P}$ is the distribution of eigenvalues, which is an $n \times n$ point process, then (**) represents the number of eigenvalues located in A. Therefore, it is important to have the exact formula for 1-point correlation function.

Summary

Def. 7.5 (k-point correlation function): The k-point correlation function ρ_k of a point process is a function of k variables, s.t. for distinct points $x_1, \cdots, x_k$, $\rho_k(x_1, \cdots, x_k)dx_1 \cdots dx_k$ is equal to the probability to have a point in each of the infinitestimal intervals $[x_j, x_j+dx_j]$, $j=1,2,\cdots,k$.

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Further Reading:

We can generalize a one-matrix model to two-matrix model, and define a (two-level) ~~probability~~ random point field on $X = \mathbb{R} \sqcup \mathbb{R}$ of the form $P_{(r,s)}(x_1, \cdots, x_r; y_1, \cdots, y_s)$, see, for example:

① A. Soshnikov. Determinantal random point fields. Usp. Mat. Nauk. 55 (2000), 107–160.

② M. Bertola et al., Cauchy-Laguerre two-matrix model and the Meijer-G random point field. Comm. Math. Phys., 326 (2014), 111–144.

Notes

For k-point correlation function, we have

$$\int_{A_1}\cdots\int_{A_k} P_k(x_1, \cdots, x_k)\,dx_1\cdots dx_k = \mathbb{E}\Big[\prod_{j=1}^{k} \#(\chi \cap A_j)\Big],$$

where $\{A_j\}_{j=1}^{k}$ are k disjoint Borel sets.

If A_j are not disjoint, (e.g. if $A_j = A$ for every j), then

$$\frac{1}{k!}\int_{A^k} P_k(x_1, \cdots, x_k)\,dx_1\cdots dx_k = \mathbb{E}\left[\binom{\#(\chi \cap A)}{k}\right].$$

<u>Lemma 7.6</u> (An explicit formula for k-point correlation function): If $p(x_1, \cdots, x_n)$ is a symmetric probability density on $\mathbb{R}^n$, then the n-point process that is associated with p has k-point correlation function

$$P_k(x_1, \cdots, x_k) = \frac{n!}{(n-k)!}\int_{\mathbb{R}^{n-k}} p(x_1, \cdots, x_n)\,dx_{k+1}\cdots dx_n.$$

That is, $\frac{(n-k)!}{n!}P_k(x_1, \cdots, x_k)$ is the marginal density of p.

<u>proof</u>: We give a proof for $k=2$, and the general case could be verified by induction. Suppose A and B are disjoint Borel sets. Let 1_A be the characteristic function of A defined by $1_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A,\end{cases}$ then

$$\mathbb{E}[\#(\chi\cap A)\cdot\#(\chi\cap B)] = \int_{\mathbb{R}^n} p(x_1, \cdots, x_n)\Big(\sum_{j=1}^{n} 1_A(x_j)\Big)\Big(\sum_{j=1}^{n} 1_B(x_j)\Big)\,dx_1\cdots dx_n$$

Summary

$$= \sum_{j,k=1}^{n}\int_{\mathbb{R}^n} p(x_1, \cdots, x_n)\,1_A(x_j)\,1_B(x_k)\,dx_1\cdots dx_n.$$

The integral vanishes when $j=k$ since A and B are disjoint. By symmetry we have

$$\mathbb{E}[\#(\chi\cap A)\cdot\#(\chi\cap B)] = n(n-1)\int_A\int_B\Big[\int_{\mathbb{R}^{n-2}} p(x_1, \cdots, x_n)\,dx_3\cdots dx_n\Big]\,dx_1dx_2.$$

Therefore. $P_2(x_1, x_2) = \frac{n!}{(n-2)!}\int_{\mathbb{R}^{n-2}} p(x_1, \cdots, x_n)\,dx_3\cdots dx_n$

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3. Gap probability

Let $\mathbb{P}_A(n) = \mathbb{P}$(there are exactly n points in A). If there are n points of the process in A, then the number of ordered k tuples in A is $\binom{n}{k}$. Thus

$$\frac{1}{k!}\int_{A^k} \rho_k(x_1,\cdots,x_k)dx_1\cdots dx_k = \sum_{n=k}^{+\infty}\binom{n}{k}\mathbb{P}_A(n),$$

while $\sum_{n=k}^{+\infty}\binom{n}{k}\mathbb{P}_A(n) = \sum_{n=0}^{+\infty}\mathbb{P}_A(n) = 1$ for $k=0$.

Therefore,

$$1+\sum_{k=1}^{+\infty}\frac{(-1)^k}{k!}\int_{A^k}\rho_k(x_1,\cdots,x_k)dx_1\cdots dx_k = \sum_{k=0}^{+\infty}\sum_{n=k}^{+\infty}(-1)^k\binom{n}{k}\mathbb{P}_A(n)$$

$$= \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}\right)\mathbb{P}_A(n)$$

Noting that $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = (1-1)^n = \delta_{n,0} = \begin{cases}1, & \text{if } n=0\\ 0, & \text{if } n\geq 1\end{cases}$

we have $\mathbb{P}_A(0) = 1+\sum_{k=1}^{+\infty}\frac{(-1)^k}{k!}\int_{A^k}\rho_k(x_1,\cdots,x_k)dx_1\cdots dx_k$, which means the probability that there is no point in A. This is the so-called gap probability.

E.x: Write down the Fredholm determinant expression for the probability $\mathbb{P}(\lambda_{max}\leq a)$ and $\mathbb{P}(\lambda_{min}\geq a)$.

Remark: If $\mathbb{P}$ is the distribution of eigenvalues, then the gap probability means the probability that there is no eigenvalue in A.

Summary