No.
Date.

Cues

Lecture 6.

Eigenvalue distribution of unitary invariant ensembles

probability measures on random matrices

↓

probability measures on the eigenvalues

① the space of eigenvalues

② what's the relation between probability measures on M and probability measure on $(\lambda_1, \cdots, \lambda_n)$.

Notes

1. The space of eigenvalues

We start with a general invariant ensemble with potential V. Since M is Hermitian, eigenvalues of M are real. We can consider eigenvalues $\lambda_1, \cdots, \lambda_n$ of M in 2 ways. ① They are in an increasing order $\lambda_1 \leq \cdots \leq \lambda_n$, i.e. $(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n_{\leq}$; ② They are in an arbitrary unordered fashion, i.e. $(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$.

In both cases, we can claim that eigenvalues will be distinct almost surely, and define probability density for eigenvalues on $\mathbb{R}^n_{\leq}$ as well as $\mathbb{R}^n$. Suppose $p(\lambda_1, \cdots, \lambda_n)$ is the probability density on $\mathbb{R}^n_{\leq}$ for the ordered eigenvalues. Then for an unordered tuple $(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ of distinct eigenvalues, we have $\lambda_i \neq \lambda_j$ if $i \neq j$ a.s.. Therefore, there is a unique permutation $\sigma \in S_n$, s.t. $\lambda_{\sigma(1)} < \cdots < \lambda_{\sigma(n)}$.

Then we define the symmetrization of p by

$$p^{sym}(\lambda_1, \cdots, \lambda_n) = \frac{1}{n!} p(\lambda_{\sigma(1)}, \cdots, \lambda_{\sigma(n)}), \quad (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n.$$

Conversely, we have

$$p(\lambda_1, \cdots, \lambda_n) = n!\, p^{sym}(\lambda_1, \cdots, \lambda_n), \quad (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n_{\leq}.$$

Summary

Note that $\mathbb{R}^n$ is a symmetric space, which is easier for analysis. Therefore, we prefer to consider unordered eigenvalues.

No.

Date.

Cues | Notes

2. From the space of Hermitian matrices to the space of eigenvalues

Since M is Hermitian, it is known that we have the following spectral decomposition $M = U^*\Lambda U$, where Λ is a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ and U is an unitary matrix. Therefore, we want to calculate the relationship between dM and $d\Lambda$.

<u>Prop. 6.1</u> (Change of variables):

For an $n\times n$ Hermitian matrix M, we have

$$dM = C_n \prod_{1\le i<j\le n} (\lambda_j - \lambda_i)^2 \, d\lambda_1 \cdots d\lambda_n \, dU,$$

where $\{\lambda_i\}_{i=1}^n$ are eigenvalues of M, C_n is a constant, and dU is the Haar measure on the unitary group $U(n)$.

E.x.1: Prove $A^* dM A = (\det(A^*A))^n dM$ by using elementary matrices, which satisfy

$E(j\to k)^* dM E(j\to k) = dM$

$E(j\to j+k)^* dM E(j\to j+k) = dM$

$E(j\to \alpha j)^* dM E(j \to \alpha j) = |\alpha|^{2n} dM$

<u>proof</u>: We first claim that for any $n\times n$ nonsingular matrix A, we have $A^* dMA = (\det(A^*A))^n dM$.

Then from $M = U^*\Lambda U$, we have $dM = dU^*\Lambda U + U^* d\Lambda U + U^*\Lambda dU$, from which we could get

$$dM = UdU^*\Lambda - \Lambda U dU^* + d\Lambda = d\Lambda + [UdU^*, \Lambda].$$

With the notation $\vec{u}_k = (u_{k1}, \cdots, u_{kn})$ for the k^{th} row of U, we have $UdU^* = \begin{pmatrix} \vec{u}_1 \\ \vdots \\ \vec{u}_n \end{pmatrix} (d\vec{u}_1^*, \cdots, d\vec{u}_n^*) = (\vec{u}_i d\vec{u}_j^*)_{i,j=1}^n$.

Summary

Therefore, $[UdU^*, \Lambda] = \left((\lambda_j - \lambda_i) \vec{u}_i d\vec{u}_j^* \right)_{i,j=1}^n$, and

$$dM = \begin{pmatrix} d\lambda_1 & (\lambda_2-\lambda_1)\vec{u}_1 d\vec{u}_2^* & \cdots & (\lambda_n - \lambda_1)\vec{u}_1 d\vec{u}_n^* \\ & d\lambda_2 & \cdots & (\lambda_n - \lambda_2)\vec{u}_2 d\vec{u}_n^* \\ & & \ddots & \vdots \\ & & & d\lambda_n \end{pmatrix}.$$

Since dM is Hermitian, we know that

$$dM = d\lambda_1 \cdots d\lambda_n \prod_{1\le i<j\le n} (\lambda_j - \lambda_i)^2 \mathrm{Re}(\vec{u}_i d\vec{u}_j^*) \mathrm{Im}(\vec{u}_i d\vec{u}_j^*)$$

#

No.
Date.

Cues

E.x.2:

For an $n\times n$ symmetric matrix M, show that

$dM=\tilde{C}_n\prod_{1\le i<j\le n}|\lambda_j-\lambda_i|\,d\lambda_1\cdots d\lambda_n\times d\tilde{U}$,

where $d\tilde{U}$ is the Haar measure on the orthogonal group $O(n)$.

Ref.:《华罗庚文集:代数卷》

Notes

An alternative proof of this theorem was given by "Fyodorov Y., Introduction to the random matrix theory: Gaussian Unitary Ensemble and beyond. arXiv: math-ph/0412017" in terms of Riemann metric and volume.

3. The Weyl integration formula

A function $f:\mathcal{H}_n\to\mathbb{C}$ is a class function if $f(U^*MU)=f(M)$ for all $U\in U(n)$. This means that $f(M)$ depends only on the eigenvalues of M. If $\lambda_1,\cdots,\lambda_n$ are the eigenvalues of M and f is a class function, then we could also write $f(M)=g(\lambda_1,\cdots,\lambda_n)$, where g is a symmetric function of $\lambda_1,\cdots,\lambda_n$.

Thm. 6.2 (Weyl integration formula):

For an integrable class function f on $\mathcal{H}_n$, we have

$$\int_{\mathcal{H}_n} f(M)\,dM = C_n\int_{\mathbb{R}^n} g(\lambda_1,\cdots,\lambda_n)\prod_{1\le i<j\le n}(\lambda_j-\lambda_i)^2\,d\lambda_1\cdots d\lambda_n$$

where C_n is a constant.

This proof is similar to Prop. 6.1 by using a change of variables. In fact, the so-called Weyl integration formula could be applied to all classical groups.

Summary

Applying the Weyl integration formula to random matrix ensemble $\frac{1}{Z_n}e^{-\mathrm{Tr}V(M)}dM$, we find that if $\lambda_1,\cdots,\lambda_n$ are eigenvalues of M, then $\mathrm{Tr}V(M)=\sum_{j=1}^n V(\lambda_j)$, and so $e^{-\mathrm{Tr}V(M)}$ is a class function. Therefore, we have

$$\mathbb{E}[f]=\frac{1}{Z_n}\int_{\mathcal{H}_n} f(M)e^{-\mathrm{Tr}V(M)}dM=\frac{C_n}{Z_n}\int_{\mathbb{R}^n} g(\lambda_1,\cdots,\lambda_n)\prod_{1\le i<j\le n}(\lambda_j-\lambda_i)^2\prod_{j=1}^n e^{-V(\lambda_j)}d\lambda_1\cdots d\lambda_n$$

No.
Date.

Cues

- $\tilde{Z}_n$ is related to Z_n via the formula
$$\tilde{Z}_n = \frac{Z_n}{C_n} = \pi^{-\frac{n(n+1)}{2}} \left(\prod_{j=1}^{n} j!\right) Z_n$$
and the computation of C_n could be found at the reading material.

Notes

To conclude, we have the following theorem.

Thm. 6.3 (j.p.d.f. for eigenvalues):

Let $P(M)dM = \frac{1}{Z_n} e^{-\mathrm{Tr}\, V(M)} dM$ be an unitary invariant ensemble. Then the unordered eigenvalues $\lambda_1, \cdots, \lambda_n$ of M have the joint probability density function

$$\frac{1}{\tilde{Z}_n} \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i)^2 \prod_{i=1}^{n} e^{-V(\lambda_i)} \qquad (*)$$

with respect to Lebesgue measure on $\mathbb{R}^n$, where $\tilde{Z}_n$ is a normalization factor.

The eigenvalue distribution is related to orthogonal polynomials with weight e^{-V} on $\mathbb{R}$. Therefore, we sometimes call $(*)$ as an orthogonal polynomial ensemble. We have some direct corollary from orthogonal polynomial theory.

Corollary 6.4 (About the partition function / normalization):

We have $\mathbb{E}[1] = \tilde{Z}_n = n!\, D_n$, where D_n is the Hankel determinant $D_n = \det(m_{j+k-2})_{j,k=1}^{n}$ with $m_k = \int_{\mathbb{R}} x^k e^{-V(x)} dx$

Summary

Corollary 6.5 (About the average characteristic polynomials):

We have $\mathbb{E}[\det(xI_n - M)] = P_n(x)$, which is equal to the monic orthogonal polynomials of degree n with weight $e^{-V(x)}$.

$\Rightarrow$ This is the reason why we call the previous examples as LUE & JUE.