No.
Date.

Cues

Lecture 5.

Random matrix models & invariant ensemble

Random matrix model is a finite approximation of an infinite dimensional random operator such as the Hamiltonian of heavy nuclei, and the eigenvalue of random matrices is an approximation of energy levels. This is the reason why we're interested in:

① the distribution of eigenvalues

② the limiting behaviour of eigenvalues when the size of matrix tends to infinity.

Notes

1. An introduction to random matrix models

$$\text{Random matrix models} \begin{cases} \text{space of matrices} \\ \text{Prob. measure on this space} \end{cases}$$

There are two main classes of random matrices:

① Wigner ensemble. The entries of the matrices are independent random variables.

② Invariant ensemble. The prob. measure is invariant under the action of a group, which reflects a priori symmetry in the model.

Usually, one is typically interested in the eigenvalues of random matrices, so the invariant ensemble is easy for us to consider the distribution of eigenvalues according to the symmetry. This course is focused on unitary invariant ensembles of Hermitian matrices, as they are mostly closed to orthogonal polynomials.

2. Unitary invariant ensemble of Hermitian matrices

Let's denote $\mathcal{H}_n = \{M \mid M \text{ is an } n\times n \text{ Hermitian matrix}\}$.

The Lebesgue measure on $\mathcal{H}_n$ is given by $dM = \prod_{i=1}^{n} dM_{ii} \times$

Summary

$\prod_{1\le i<j\le n} d\mathrm{Re}M_{ij}\, d\mathrm{Im}M_{ij}$. The invariant ensemble is a probability measure on $\mathcal{H}_n$ that is invariant under unitary conjugation $M \to U^*MU$, $\forall U \in U(n)$.

Remark 5.1: For a probability measure $p(M)dM$, since $d(U^*MU) = dM$ always holds true, we know that if $p(M)dM$ is unitarily invariant, then we need $p(U^*MU)=p$

No.

Date.

Cues

- In fact, for a polynomial $V(x)=\sum_{j=0}^{n} c_j x^j$, we write $V(M)=\sum_{j=0}^{n} c_j M^j$. Obviously, we have

$$\mathrm{Tr}(V(M)) = \mathrm{Tr}(V(U^*MU)).$$

For general V, we can define $V(M)$ by matrix calculus.

Further reading:

"Analytic function of a matrix" in wikipedia.

Notes

<u>Def. 5.2</u> (Unitary invariant ensemble of Hermitian matrices):

An unitary invariant ensemble of $n\times n$ Hermitian matrices is a probability measure on $\mathcal{H}_n$, given by

$$p(M)\,dM = \frac{1}{Z_n} e^{-\mathrm{Tr}\,V(M)}\,dM,$$

where $V:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ is a potential function and Z_n is a normalization (partition function)

$$Z_n = \int_{\mathcal{H}_n} e^{-\mathrm{Tr}\,V(M)}\,dM.$$

It should be noted that the partition function should be finite, which means that the function V needs to have enough increase at $\pm\infty$ to guarantee that the integral converges. A typical condition is $\lim\limits_{x\to\pm\infty} \dfrac{V(x)}{\log(1+x^2)} = \infty$.

3. Main examples

In this part, we introduce three main examples related to classic orthogonal polynomials.

Example 1. (Gaussian Unitary Ensemble, GUE).

<u>Def. 5.3</u>: The GUE is the probability measure $\frac{1}{Z_n} e^{-\mathrm{Tr}M^2}\,dM$ on $\mathcal{H}_n$. It is unitary invariant ensemble with quadratic potential $V(x)=x^2$.

Summary

In the case of GUE, we can write down the probability measure in terms of matrix entries. Note that

$$\mathrm{Tr}(M^2) = \sum_{i=1}^{n}(M^2)_{ii} = \sum_{i,j=1}^{n} M_{ij}M_{ji} = \sum_{i=1}^{n} M_{ii}^2 + \sum_{i\neq j} M_{ij}M_{ji} = \sum_{i=1}^{n} M_{ii}^2 + 2\sum_{1\le i<j\le n}\left[(\mathrm{Re}\,M_{ij})^2 + (\mathrm{Im}\,M\right.$$

then we have

$$\frac{1}{Z_n} e^{-\mathrm{Tr}M^2}\,dM = \frac{1}{Z_n}\prod_{i=1}^{n} e^{-M_{ii}^2}\cdot\prod_{1\le i<j\le n} e^{-2(\mathrm{Re}\,M_{ij})^2}\cdot e^{-2(\mathrm{Im}\,M_{ij})^2}\,dM. \quad (*)$$

Cues

E.x.1.:

Show that for any quadratic potential $V(x)=ax^2+bx+c$, $a>0$, the invariant ensemble $\frac{1}{Z_n}e^{-\mathrm{Tr}V(M)}$ can be reduced to the GUE by an affine change of variables $M \to pM+q$ for certain p and q.

To take V(x) into $e^{-\mathrm{Tr}V(M)}$, we need to use Jacobi's formula on the derivative of determinants. Firstly, we have

$\frac{d}{dt}\det A(t) = \det A(t) \cdot \mathrm{tr}(A^{-1}\frac{dA}{dt})$, then by taking $A(t)=e^{tB}$, we get $\frac{d}{dt}\det e^{tB} = \det e^{tB} \cdot \mathrm{tr} B$,

Notes

Recall that dM is the Lebesgue measure

$$dM = \prod_{i=1}^{n} dM_{ii} \prod_{1\le i<j\le n} d(\mathrm{Re}M_{ij})\, d(\mathrm{Im}M_{ij}),$$

and moreover, $\frac{1}{\sqrt{2\pi\sigma}}\exp(-\frac{(x-\mu)^2}{2\sigma^2})$ is the density of normal dist. with mean μ and variance σ^2. Therefore, (*) is the joint density function with $n+\frac{n(n-1)}{2}\times 2 = n^2$ independent random variables, where $M_{ii}\sim N(0,\frac{1}{2})$, $\mathrm{Re}M_{ij}\sim N(0,\frac{1}{4})$, $\mathrm{Im}M_{ij}\sim N(0,\frac{1}{4})$.

<u>Proposition 5.4.</u> $Z_n^{GUE} = \pi^{n^2/2}\cdot 2^{-n(n-1)/2}$.

<u>proof:</u> In the case of GUE, we have

$$Z_n = \int_{\mathcal{H}_n} e^{-\mathrm{Tr}M^2} dM = \int_{\mathbb{R}^{n^2}} \prod_{i=1}^{n} e^{-M_{ii}^2} \prod_{1\le i<j\le n} e^{-2(\mathrm{Re}M_{ij})^2} e^{-2(\mathrm{Im}M_{ij})^2} dM$$

$$= \prod_{i=1}^{n}\sqrt{\pi} \times \prod_{1\le i<j\le n}\sqrt{\frac{\pi}{2}} = \pi^{n^2/2}\cdot 2^{-\frac{n(n-1)}{2}}. \qquad \#$$

Example 2. (Laguerre Unitary Ensemble, LUE)

The LUE with $\alpha>-1$ is the probability measure on the space of $n\times n$ positive definite Hermitian matrices given by

$$\frac{1}{Z_n}(\det M)^\alpha e^{-\mathrm{Tr}M}\, dM.$$

The LUE is an unitary invariant ensemble with potential

$$V(x) = \begin{cases} x-\alpha\log x, & x>0 \\ +\infty, & x\le 0 \end{cases}$$

Next, we show how to construct LUE:

If X is a random matrix with independent entries, then X^*X is called a Wishart matrix. The LUE arises as Wishart matrix as follows:

X: $m\times n$ complex random matrix ($m\ge n$).

$\mathrm{Re}X_{ij}$, $\mathrm{Im}X_{ij}$ i.i.d $\sim N(0,1)$, then $M=XX^*$ is an LUE random matrix with parameter $\alpha = m-n$.

Summary

which shows $\det e^B = e^{\mathrm{tr}B}$.

Therefore, $e^{+\alpha\mathrm{Tr}\log M} = \det(e^{\log M^\alpha}) = \det M^\alpha$.

No.

Date.

Cues

Notes

The proof of this construction is a little bit tricky. One can refer:

"M. Ghosh and B. Sinka. A simpler derivation of the Wishart distribution. The American Statistician, 2002."

Example 3. (Jacobi Unitary Ensemble, JUE)

A Hermitian positive definite contraction matrix is a Hermitian matrix with all eigenvalues in $[0,1]$.

<u>Def. 5.5</u>: The JUE with parameters $\alpha > -1, \beta > -1$ is the probability measure on the space of $n \times n$ Hermitian positive definite contraction matrices given by

$$\frac{1}{Z_n} (\det M)^{\alpha} \det(I-M)^{\beta} dM.$$

This is an unitary invariant ensemble with potential

$$V(x) = \begin{cases} -\beta \log (1-x) - \alpha \log x, & 0 < x < 1, \\ +\infty & , \text{elsewhere}. \end{cases}$$

There are different methods to construct JUE. One example comes from truncations of random unitary matrices. Suppose that U is an unitary matrix of size N, and it is random according to Haar measure on the unitary group. Let T be the left upper submatrix of U of size $m \times n$, then $T^* T$ is a JUE with parameter $\alpha = N-m-n$ and $\beta = m-n$.

Summary