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Lecture 4.

A specific example of classic orthogonal polynomials — Hermite polynomials

$$\int_{-\infty}^{+\infty}\left(\frac{d^n}{dx^n}e^{-x^2}\right)H_m(x)\,dx = \left(\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}\right)H_m(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\left(\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}\right)\frac{d}{dx}H_m(x)\,dx$$
$$= \cdots = (-1)^n\int_{-\infty}^{+\infty}e^{-x^2}\frac{d^n}{dx^n}H_m(x)\,dx$$

Notes

1. Definition and orthogonality

Hermite polynomials are given by the Rodrigues formula $e^{-x^2}H_n(x) = (-1)^n\frac{d^n}{dx^n}(e^{-x^2})$.

Easily, we can get

$H_0(x)=1,\ H_1(x)=2x,\ H_2(x)=4x^2-2,\ H_3(x)=8x^3-12x,$ ......

Moreover, we have the following orthogonality for Hermite polynomials.

Proposition 4.1 (Orthogonality for Hermite polynomials): Hermite polynomials satisfy the following relation

$$\int_{-\infty}^{+\infty}H_n(x)H_m(x)e^{-x^2}dx = \sqrt{\pi}\cdot 2^n\cdot n!\,\delta_{nm}.$$

proof: By using Rodrigues formula, we have

$$\int_{-\infty}^{+\infty}H_n(x)H_m(x)e^{-x^2}dx = \int_{-\infty}^{+\infty}(-1)^n\left(\frac{d^n}{dx^n}e^{-x^2}\right)H_m(x)dx$$
$$= \int_{-\infty}^{+\infty}e^{-x^2}\left(\frac{d^n}{dx^n}H_m(x)\right)dx = 0 \text{ if } n>m.$$ Moreover, by noting that $H_n(x) = 2^nx^n + \text{l.o.t.}$, it is known that

$$\int_{-\infty}^{+\infty}(H_n(x))^2e^{-x^2}dx = \int_{-\infty}^{+\infty}e^{-x^2}\left(\frac{d^n}{dx^n}H_n(x)\right)dx = \int_{-\infty}^{+\infty}e^{-x^2}2^n\cdot n!\,d\,[x]$$
$$= \sqrt{\pi}\cdot 2^n\cdot n! \quad \#$$

Summary

Remark 4.2: From the orthogonality, it is known that $\frac{D_{n+1}}{D_n} = 2^{-n}\sqrt{\pi}\cdot n!$, and $D_1 = m_0 = \sqrt{\pi}$. Thus, we have $D_{n+1} = \frac{D_{n+1}}{D_n}\cdot\frac{D_n}{D_{n-1}}\cdots\frac{D_2}{D_1}\cdot D_1 = \prod_{k=0}^{n}2^{-k}\sqrt{\pi}\cdot k!$

$= 2^{-\frac{n(n+1)}{2}}\cdot\pi^{\frac{n+1}{2}}\cdot\prod_{k=0}^{n}k!$, which gives an exact evaluation for Hankel determinant

$$\det\left(\int_{-\infty}^{+\infty}x^{i+j-2}e^{-x^2}dx\right)_{i,j=1}^{n} = 2^{-\frac{n(n-1)}{2}}\pi^{\frac{n}{2}}\cdot\prod_{k=0}^{n-1}k!.$$

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2. Recurrence relations for Hermite polynomials

From the Rodrigues formula $e^{-x^2}H_n(x) = (-1)^n \frac{d^n}{dx^n}(e^{-x^2})$, we have

$$e^{-x^2}H_n(x) = (-1)^n \frac{d^{n-1}}{dx^{n-1}}\left(-2xe^{-x^2}\right)$$

$$= (-1)^n\left[-2x\frac{d^{n-1}}{dx^{n-1}}(e^{-x^2}) - 2(n-1)\frac{d^{n-2}}{dx^{n-2}}(e^{-x^2})\right]$$

$$= \left(2xH_{n-1}(x) - 2(n-1)H_{n-2}(x)\right)e^{-x^2}.$$

Leibniz rule:

$$\frac{d^n}{dx^n}(f(x)g(x)) = \sum_{k=0}^{n}\binom{n}{k}f^{(k)}(x)\cdot g^{(n-k)}(x)$$

This is the 3-term recurrence relation

$$2xH_n(x) = H_{n+1}(x) + 2nH_{n-1}(x) \qquad (*)$$

for Hermite polynomials.

On the other hand, if we take the derivative of Rodrigues formula, then we have $\frac{d}{dx}(e^{-x^2}H_n(x)) = -e^{-x^2}H_{n+1}(x)$, which gives $H_{n+1}(x) = 2xH_n(x) - H_n'(x)$. By putting into equation (*), we get $H_n'(x) = 2nH_{n-1}(x)$, and moreover, $H_n''(x) = 2nH_{n-1}'(x) = 2n\left[2xH_{n-1}(x) - H_n(x)\right] = 2xH_n'(x) - 2nH_n(x)$. Therefore, $H_n(x)$ is a solution of 2^{nd} order differential equation $y''(x) - 2xy'(x) + 2ny(x) = 0$. This equation is equivalent to the Schrödinger equation for the harmonic oscillator.

E.x. 1. Using the 2^{nd} order differential equation to show that $H_n(x) = n!\sum_{j=0}^{\lfloor n/2\rfloor}\frac{(-1)^j(2x)^{n-2j}}{j!(n-2j)!}$.

Summary

Hint: Assuming that $H_n(x) = 2^n x^n + a_{n-1}x^{n-1} + \cdots + a_0x^0$, then by taking it into the 2^{nd} DE, we have

$$\frac{d^2}{dx^2}(2^nx^n + a_{n-1}x^{n-1} + \cdots + a_0x^0) = 2x\frac{d}{dx}(2^nx^n + a_{n-1}x^{n-1} + \cdots + a_0x^0) - 2n(2^nx^n + a_{n-1}x^{n-1} + \cdots + a_0x^0)$$

$\Rightarrow a_{n-1} = 0,\ a_{n-2} = 2^{n-2}\cdot n(n-1),\ a_{n-4} = \frac{1}{2}\cdot 2^{n-4}n(n-1)(n-2)(n-3), \cdots\ H_{2n}(x) = \sum_{j=0}^{n}\frac{(2n)!(-1)^j}{j!(2n-2j)!}(2x)^{2n-2j}$

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E.x.2. According to the explicit formula of Hermite polynomials, show the equivalent integral formula

$$H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{+\infty} (x+iy)^n e^{-y^2} dy.$$

3. Generating function and integral representation

In fact, according to the Rodrigues formula, we have

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) = (-1)^n e^{x^2} \frac{n!}{2\pi i} \oint_{C_\infty} \frac{e^{-t^2} dt}{(t-x)^{n+1}}.$$

By using this formula, we have the generating function

$$\sum_{n=0}^{+\infty} \frac{H_n(x)}{n!} z^n = \frac{e^{x^2}}{2\pi i} \oint_{C_\infty} e^{-t^2} \sum_{n=0}^{+\infty} (-1)^n \frac{z^n}{(t-x)^{n+1}} dt$$

$$= \frac{e^{x^2}}{2\pi i} \oint_{C_\infty} \frac{e^{-t^2} dt}{t-x+z} = e^{x^2} \cdot e^{-(x-z)^2} = e^{2xz-z^2},$$

which converges for every x and z in the complex plane.

Moreover, from this generating function, we can get a contour integral representation for Hermite polynomials

$$H_n(x) = \frac{n!}{2\pi i} \oint_{C_0} \frac{e^{2xz-z^2}}{z^{n+1}} dz.$$

E.x.3. Using this method to show

$$\sum_{n=0}^{+\infty} \frac{H_{n+k}(x)}{n!} z^n = \exp(2xz-z^2) H_k(x-z).$$

Summary

E.x.4. For Laguerre polynomials, show the 3-term recurrence relation, 2^{nd} order differential equation, generating function and integral representation from its Rodrigues formula.

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Note: In this example, we don't require that random variables are i.i.d., but assume that they have the same 1^{st} and 2^{nd} order moments.

Recall (Favard's theorem): If a family of polynomials satisfy a 3-term recurrence relation, then these polynomials are orthogonal w.r.t. some measure.

Notes

4. A first encounter with random matrices

M_n: $n\times n$ Hermitian random matrix

$$(M_n)_{k,l} = \begin{cases} X_{k,l} + iY_{k,l}, & k<l \\ X_{k,l} - iY_{k,l}, & k>l \\ X_{kk}, & k=l \end{cases}$$

where $X_{k,l}, Y_{k,l}$ $(1\le k<l\le n)$ and X_{kk} $(1\le k\le n)$ are independent random variables with $\mathbb{E}(X_{kl}) = \mathbb{E}(Y_{kl}) = \mathbb{E}(X_{kk}) = 0$, and $\mathbb{E}(X_{k,l}^2) = \mathbb{E}(Y_{k,l}^2) = \mathbb{E}(X_{kk}^2) = \sigma^2 > 0$,

We are interested in the eigenvalues of this random matrix. To this end, we can consider the characteristic polynomial $\det(xI_n - M_n)$. This polynomial is random, but we can compute its expected value $P_n(x) = \mathbb{E}[\det(xI_n - M_n)]$. We call it as "average characteristic polynomial".

Thm. 4.2. The average characteristic polynomials satisfy the recurrence relation

$$P_n(x) = xP_{n-1}(x) - 2(n-1)\sigma^2 P_{n-2}(x).$$

proof: Expand the determinant along the last row to find

$$P_n(x) = \mathbb{E}\left(\sum_{l=1}^{n-1}(-1)^{n+l}(-M_n)_{n,l}\det(xI_n - M_n)_{[n]}^{[l]}\right) + \mathbb{E}\left((x-(M_n)_{n,n})\det(xI_n - M_n)_{[n]}^{[n]}\right),$$

Summary

where $A_{[n]}^{[k]}$ is the matrix A with row n and column k deleted. For the last term, we see that $(M_n)_{n,n} = X_{nn}$ is independent of all the entries in $(xI_n - M_n)_{[n]}^{[n]}$ and hence $\mathbb{E}((x-(M_n)_{n,n})\det(xI_n - M_n)_{[n]}^{[n]}) = (x - \mathbb{E}(X_{n,n}))\mathbb{E}(\det(xI_n - M_n)_{[n]}^{[n]}) = xP_{n-1}(x)$.

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Note: In fact, we don't use $\mathbb{E}[X_{k,k}^2] = \sigma^2$ in the proof. In other words, the 2^{nd} order moment for diagonals is not necessary in the proof.

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Moreover, expand $\det(xI_n - M_n)^{[\ell]}_{[n]}$ along its last column to find

$$\mathbb{E}\left[-(M_n)_{n,\ell}\det(xI_n - M_n)^{[\ell]}_{[n]}\right]$$

$$= (-1)^{n+\ell-1}\mathbb{E}\left[(M_n)_{n,\ell}(M_n)_{\ell,n}\right] \cancel{\det}\, \mathbb{E}\left[\det(xI_n - M_n)^{[n,\ell]}_{[n,\ell]}\right]$$

$$+ \sum_{k\neq\ell}(-1)^{n+k-1}\mathbb{E}\left[(M_n)_{n,\ell}\right]\mathbb{E}\left[(M_n)_{k,n}\right]\mathbb{E}\left[\det(xI_n - M_n)^{[n,\ell]}_{[n,k]}\right]$$

$$= (-1)^{n+\ell-1}\cdot 2\sigma^2 P_{n-2}(x). \qquad \#$$

We can easily find the initial values $P_0(x) = 1$ and $P_1(x) = x$. Moreover, we have the following corollary.

<u>Corollary 4.3</u>: The average characteristic polynomial is a scaled Hermite polynomial $P_n(x) = \sigma^n H_n(x/2\sigma)$, where $H_n(x)$ is the Hermite polynomial of degree n.

Summary