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Cues

Lecture 3.

Integral representation of orthogonal polynomials, and examples of classic orthogonal polynomials

- This proposition says the integral over determinant × determinant is still a determinant.

Notes

1. Andréief formula

Proposition 3.1 (Andréief formula)

If $\{\varphi_i(x)\}_{i=1,2,\cdots,n}$ and $\{\psi_i(x)\}_{i=1,2,\cdots,n}$ are square integrable functions with respect to a measure $d\mu$, then we have

$$\frac{1}{n!}\int_{\mathbb{R}^n} \det(\varphi_i(x_j))_{i,j=1}^n \det(\psi_i(x_j))_{i,j=1}^n \prod_{i=1}^n d\mu(x_i)$$
$$= \det\left(\int_{\mathbb{R}} \varphi_i(x)\psi_j(x)\, d\mu(x)\right)_{i,j=1}^n.$$

proof: Noting that

$$\int_{\mathbb{R}^n} \det(\varphi_i(x_j))_{i,j=1}^n \det(\psi_i(x_j))_{i,j=1}^n \prod_{i=1}^n d\mu(x_i)$$
$$= \sum_{\pi\in S_n} (-1)^\pi \int_{\mathbb{R}^n} \prod_{i=1}^n \varphi_i(x_{\pi(i)}) \det(\psi_i(x_j))_{i,j=1}^n \prod_{i=1}^n d\mu(x_i)$$
$$= \sum_{\pi\in S_n} (-1)^\pi \int_{\mathbb{R}^n} \det(\varphi_{\pi^{-1}(j)}(x_j)\psi_i(x_j))_{i,j=1}^n \prod_{i=1}^n d\mu(x_i)$$
$$= \sum_{\pi\in S_n} (-1)^\pi \det\left(\int_{\mathbb{R}} \varphi_{\pi^{-1}(j)}(x)\psi_i(x)\, d\mu(x)\right)_{i,j=1}^n$$
$$= \sum_{\pi\in S_n} (-1)^\pi\cdot(-1)^\pi \det\left(\int_{\mathbb{R}} \varphi_j(x)\psi_i(x)\, d\mu(x)\right)_{i,j=1}^n,$$

which is equal to $n!\det\left(\int_{\mathbb{R}} \varphi_i(x)\psi_j(x)\, d\mu(x)\right)_{i,j=1}^n$. #

Summary

Ex. 1: If $\{\varphi_i(x)\}_{i=1,2,\cdots,n}$ belong to $L^2(d\mu_1)$, $\{\psi_i(x)\}_{i=1,2,\cdots,n}$ belong to $L^2(d\mu_2)$, and $K(x,y)$: ~~$\mathbb{R}\times\mathbb{R}$~~ $\to \mathbb{R}$ is an analytic function, then show that

$$\frac{1}{(n!)^2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \det(\varphi_i(x_j))_{i,j=1}^n \det(\psi_i(y_j))_{i,j=1}^n \det(K(x_i,y_j))_{i,j=1}^n \prod_{i=1}^n d\mu_1(x_i)\,d\mu_2(y_i)$$
$$= \det\left(\int_{\mathbb{R}\times\mathbb{R}} \varphi_i(x)K(x,y)\psi_j(y)\,d\mu_1(x)\,d\mu_2(y)\right)_{i,j=1}^n.$$

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2. Applications into orthogonal polynomials

In Andréief formula, if we take $\varphi_i(x) = \psi_i(x) = x^{i-1}$, then

$$D_n = \det\left(m_{i+j-2}\right)_{i,j=1}^{n} = \det\left(\int_{\mathbb{R}} x^{i+j-2}\, d\mu(x)\right)_{i,j=1}^{n}$$
$$= \frac{1}{n!}\int_{\mathbb{R}^n} \prod_{1\le i<j\le n} (x_j - x_i)^2 \prod_{i=1}^{n} d\mu(x_i).$$

Moreover, we have the following corollary.

Corollary 3.2:

If μ is a probability measure, then D_n is always positive.

In this case, orthogonal polynomials induced by μ always exist and are unique.

- μ can be some continuous measures, as well as some discrete measures. Therefore, we have continuous orthogonal polynomials, and discrete ~~continuous~~ orthogonal poly.

Probabilistic interpretation: If μ is absolutely continuous with respect to Lebesgue measure, i.e. $d\mu(x) = w(x)\,dx$, then

$$\frac{1}{n!\,D_n} \prod_{1\le i<j\le n} (x_j - x_i)^2 \prod_{i=1}^{n} w(x_i)\,dx_i$$

is the density of a probability measure on $\mathbb{R}^n$. This density is an example of an interacting particle system, where the total energy is given by $e^{-V(x_1,\cdots,x_n)}$, and

$$V(x_1,\cdots,x_n) = -2\sum_{i\neq j}\log|x_j - x_i| + \sum_{i=1}^{n} V(x_i).$$

Summary

Proposition 3.3 (Integral representation of orthogonal polynomials):

The monic orthogonal polynomial satisfies the following formula

$$P_n(x) = \frac{1}{n!\,D_n}\int_{\mathbb{R}^n} \prod_{1\le i<j\le n} (x_j - x_i)^2 \prod_{i=1}^{n} (x - x_i)\, d\mu(x_i).$$

E.x.2: Prove Proposition 3.3.

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Notes

3. Examples of classic orthogonal polynomials

In this part, we constrain ourselves to cases where μ is a continuous probability measure. There are 3 general cases studied by Jacobi, Laguerre, and Hermite separately.

Name	$\omega(x)$	support
Jacobi polynomial	beta dist. $(1+x)^{\alpha}(1-x)^{\beta}$, $\alpha,\beta>-1$	$(-1,1)$
Laguerre polynomial	gamma dist. $x^{\alpha}e^{-x}$, $\alpha>-1$	$(0,\infty)$
Hermite polynomial	normal dist. e^{-x^2}	$(-\infty,\infty)$

There are some other characterizations for classic orthogonal polynomials:

① The polynomial of degree n satisfies a 2^{nd} order linear ODE of the form $\sigma(x)y''+\tau(x)y'=\lambda_n y$, where σ and τ are two polynomials of degree at most 2 and of degree 1 respectively, independent of n, and λ_n is a real number;

② (Hahn's property) The derivatives $P_n'(x)$ are again orthogonal polynomials;

③ (Rodrigues formula) $\omega(x)P_n(x)=C_n\frac{d^n}{dx^n}[\omega(x)\sigma^n(x)]$, where σ is a polynomial of degree at most 2 and C_n is a normalizing constant.

Cues

- $\sigma(x)$ and $\tau(x)$ are related to $\omega(x)$.
- Orthogonal polynomials $P_n(x)$ are eigenvectors of Sturm-Liouville problem, and λ_n are corresponding eigenvalues.
- These characterizations are equivalent. For reference, see: R. Álvarez-Nodarse. On characterizations of classical polynomials.

Summary

④ (Pearson equation) $\omega(x)$ satisfies

$$\frac{d}{dx}(\sigma(x)\omega(x))=\tau(x)\omega(x).$$

⑤ $\sigma(x)P_n'(x)=a_nP_{n+1}(x)+b_nP_n(x)+C_nP_{n-1}(x)$, where a_n, b_n, and C_n are some constants.

A table relating $\omega(x)$ to $\sigma(x)$ and $\tau(x)$:

$\omega(x)$	$\sigma(x)$	$\tau(x)$
e^{-x^2}	1	$-2x$
$x^{\alpha}e^{-x}$	x	$-(x-\alpha-1)$
$(1+x)^{\alpha}(1-x)^{\beta}$	$1-x^2$	$(\beta+\alpha)+(\beta\mp\alpha+2)x$