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Cues

Lecture 1.

Orthogonal polynomials: Definition, determinant formula, and zeros

· Difference between orthonormal polynomials & orthogonal polynomials?

· Why it is required that $\gamma_n > 0$?

Notes

1. Definition

Notations:

$\mathbb{R}[x]$: the ring of polynomials with real coefficients

Def. 1.1 (Orthonormal polynomials):

We call polynomials $\{P_n(x)\}_{n\geq 0}$ a family of orthonormal polynomials w.r.t. a measure μ on $\mathbb{R}$ if

$$\int_{\mathbb{R}} P_n(x) P_m(x)\, d\mu(x) = \delta_{n,m} = \begin{cases} 1, & \text{if } n=m, \\ 0, & \text{if } n\neq m, \end{cases}$$

where $P_n(x) = \gamma_n x^n + \cdots$ is a polynomial of degree n with leading coefficient $\gamma_n > 0$.

Some basic facts on μ:

① Monomials $1, x, x^2, \cdots$ are $L^2(\mu)$

② All moments of μ (i.e. $\int x^i d\mu(x)$, $i=0,1,2,\cdots$) exist.

Summary

Remark 1.2: If we scale $P_n(x) \longmapsto K_n P_n(x) := \underline{P}_n(x)$, then $\{\underline{P}_n(x)\}_{n\geq 0}$ is still orthogonal. Namely, $\int_{\mathbb{R}} \underline{P}_n(x) \underline{P}_m(x) d\mu(x) = \frac{1}{K_n K_m} \int_{\mathbb{R}} P_n(x) P_m(x) d\mu(x) = \frac{1}{K_n K_m} \delta_{n,m}$

In other words, orthogonal polynomials are not uniquely determined by orthogonality but also determined by normalization condition.

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- Recall Gram-Schmidt process

2. A realization of orthogonal polynomials by Gram-Schmidt process

Denoting moments (of μ): $m_i = \int_{\mathbb{R}} x^i d\mu(x)$.

$P_0(x) = c_0$, where $c_0^2 \int_{\mathbb{R}} d\mu(x) = 1 \Rightarrow c_0 = \frac{1}{\sqrt{m_0}}$.

$P_1(x) = \tilde{c}_1 x + \tilde{c}_0$, and

$\int_{\mathbb{R}} P_1(x)\cdot x^0 d\mu(x) = 0 \Leftarrow \int_{\mathbb{R}} P_1(x) P_0(x) d\mu(x) = 0 \Rightarrow \tilde{c}_1 m_1 + \tilde{c}_0 m_0 = 0$,

$\int_{\mathbb{R}} P_1(x)\cdot \tilde{c}_1 x d\mu(x) = 1 \Leftarrow \int_{\mathbb{R}} P_1(x) P_1(x) d\mu(x) = 0 \Rightarrow \tilde{c}_1^2 m_2 + 2\tilde{c}_1\tilde{c}_0 m_1 + \tilde{c}_0^2 m_0 = 1$

By replacing $\tilde{c}_0 = -\frac{m_1}{m_0}\tilde{c}_1$, we have

$\tilde{c}_1 = m_0^{1/2}(m_0m_2 - m_1^2)^{-1/2}$, $\tilde{c}_0 = -m_1 m_0^{-1/2}(m_0m_2-m_1^2)^{-1/2}$.

In general, we can get all coefficients of $\{P_n(x)\}_{n\geq 0}$ if moments are given.

However, from above calculations, we know that moments should satisfy some constraints like $m_0 \neq 0$, $m_0m_2 - m_1^2 =$ … Whether do we have any general condition?

Summary

Remark 1.3: From the Gram-Schmidt orthogonality process, we know that it is possible to consider orthogonal relations

$$\int_{\mathbb{R}} P_n(x) x^i d\mu(x) = 0, \quad 0 \leq i \leq n-1, \quad \int_{\mathbb{R}} P_n(x)\cdot a_{n,n} x^n d\mu(x) = 1. \quad (*)$$

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- This is equivalent to a change of normalization condition.

3. Determinant formula (Existence & Uniqueness for OPs)

We want to get the existence and uniqueness for OPs by using the theory of linear system. However, the second equation in (*) is not linear (but quadratic!) To avoid this problem, here we consider <u>monic polynomials</u> $P_n(x) = \gamma_n^{-1} p_n(x)$. According to orthogonality, we still have

$$\int_{\mathbb{R}} P_n(x)\, x^i \, d\mu(x) = 0, \quad \text{if } 0 \le i \le n-1. \quad (**)$$

<u>prop 1.4</u> (Existence & Uniqueness for OPs):
Monic ~~p~~ orthogonal polynomials $\{P_n(x)\}_{n\ge 0}$ are uniquely determined by (*). Moreover, the existence and uniqueness of these polynomials is equivalent to

$$D_n := \det(m_{i+j})_{i,j=0}^{n-1} \neq 0, \quad n = 1, 2, \cdots$$

<u>proof</u>: By writting $P_n(x) = x^n + b_{n,n-1}x^{n-1} + \cdots + b_{n,0}$, from (**), we have

Summary

$$\begin{cases} m_n + b_{n,n-1} m_{n-1} + \cdots + b_{n,0} m_0 = 0, \\ m_{n+1} + b_{n,n-1} m_n + \cdots + b_{n,0} m_1 = 0, \\ \vdots \\ m_{2n-1} + b_{n,n-1} m_{2n-2} + \cdots + b_{n,0} m_{n-1} = 0, \end{cases} \Rightarrow \begin{pmatrix} m_0 & m_1 & \cdots & m_{n-1} \\ m_1 & m_2 & \cdots & m_n \\ \vdots & \vdots & & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-2} \end{pmatrix} \begin{pmatrix} b_{n,0} \\ b_{n,1} \\ \vdots \\ b_{n,n-1} \end{pmatrix} = - \begin{pmatrix} m_n \\ m_{n+1} \\ \vdots \\ m_{2n-1} \end{pmatrix}$$

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- In general, we have the following expansion formula for broader determinants:

$$\det\begin{pmatrix} z & y_1 & \cdots & y_n \\ x_1 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ x_n & a_{n1} & \cdots & a_{nn} \end{pmatrix} = z\cdot\det(a_{ij})_{i,j=1}^{n} - \sum_{i=1}^{n} x_i y_j \Delta_{ij},$$

where Δ_{ij} is the cofactor of a_{ij} in $A=(a_{ij})_{i,j=1}^{n}$.

Notes

Thus, the coefficients of $P_n(x)$ are uniquely determined if $D_n = \det(m_{i+j})_{i,j=0}^{n-1} \neq 0$. #

<u>Prop.</u> 1.5 (Determinant formula for OPs):

If $D_n = \det(m_{i+j})_{i,j=0}^{n-1} \neq 0$, then $P_n(x)$ has the following determinant formula

$$P_n(x) = \frac{1}{D_n}\det\begin{pmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & \vdots & & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-1} \\ 1 & x & \cdots & x^n \end{pmatrix}.$$

<u>Proof</u>: Noting that

$$P_n(x) = x^n + b_{n,n-1}x^{n-1} + \cdots + b_{n,0} = x^n + (1, x, \cdots, x^{n-1})\begin{pmatrix} b_{n,0} \\ b_{n,1} \\ \vdots \\ b_{n,n-1} \end{pmatrix}$$

$$= x^n - (1, x, \cdots, x^{n-1})\begin{pmatrix} m_0 & m_1 & \cdots & m_{n-1} \\ m_1 & m_2 & \cdots & m_n \\ \vdots & \vdots & & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-2} \end{pmatrix}^{-1}\begin{pmatrix} m_n \\ m_{n+1} \\ \vdots \\ m_{2n-1} \end{pmatrix},$$

which is the expansion of determinant formula.

E.x.: From the above determinant formula of $P_n(x)$, prove:

① $\int_{\mathbb{R}} P_n(x)\, x^i \, d\mu(x) = 0, \quad 0 \le i \le n-1;$

② $\int_{\mathbb{R}} P_n(x)\, x^n \, d\mu(x) = \frac{D_{n+1}}{D_n}$

Summary

<u>Remark 1.6</u>: If we replace $\mathcal{P}_n(x) = \gamma_n P_n(x)$, then we have

$$\int_{\mathbb{R}} P_n^2(x)\, d\mu(x) = \int_{\mathbb{R}} \gamma_n^{-2}\, \mathcal{P}_n^2(x)\, d\mu(x) = \gamma_n^{-2} = \frac{D_{n+1}}{D_n}.$$ Since $\gamma_n > 0$, we have

$$\gamma_n = \left(\frac{D_{n+1}}{D_n}\right)^{-1/2}, \text{ and } \mathcal{P}_n(x) = \frac{1}{\sqrt{D_n D_{n+1}}}\det\begin{pmatrix} m_0 & m_1 & \cdots & m_n \\ \vdots & \vdots & & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-1} \\ 1 & x & \cdots & x^n \end{pmatrix}.$$

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- $P_n(x)$ doesn't have real roots $\Longleftrightarrow$ $P_n(x) > 0$ for any $x \in \mathbb{R}$
- $P_n(x)$ doesn't have real roots with odd multiplicity $\Longleftrightarrow$ $P_n(x) \geq 0$ for all $x \in \mathbb{R}$ and $P_n(x) = 0$ for finite x.
- $\int_{\mathbb{R}} P_n(x)\, d\mu(x)$ could be written as $\int_{\mathbb{R}} P_n(x) \cdot 1\, d\mu(x)$.

Notes

4. Zeros of orthogonal polynomials

This part could be understood as an application of orthogonality.

<u>Prop.</u> 1.7: $P_n(x)$ has n simple, real roots.

<u>proof</u>: Firstly, we claim that $P_n(x)$ must have real roots with odd multiplicity. Otherwise, $P_n(x)$ doesn't change sign on $\mathbb{R}$, and thus

$$\int_{\mathbb{R}} P_n(x)\, d\mu(x) > 0,$$

which is contradicted with orthogonality.

Therefore, let's suppose that $P_n(x)$ has k distinct real zeros of odd multiplicity at $x_1, \cdots, x_k$ $(k \leq n)$. Let $Q_K(x) = \prod_{j=1}^{K} (x - x_j)$, then the real zeros of $Q_K(x) P_n(x)$ have even multiplicity, meaning that $\int_{\mathbb{R}} Q_K(x) P_n(x)\, d\mu(x) > 0$. According to orthogonali
it is known that $\deg Q_K \geq n$. Thus $k = n$, which implies that $P_n(x)$ has exactly n real zeros with multiplicity 1

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Summary