Solutions of H.W. 6:

1. ① Compute the discriminant: $B^2-4AC = 9y^2 > 0$ when $y \neq 0$.

② characteristic eq.: $\dfrac{dy}{dx} = \dfrac{B \pm \sqrt{B^2-4AC}}{2A} = \dfrac{3y \pm 3y}{2y}$

$\Rightarrow$ characteristic curves: (i) $\dfrac{dy}{dx} = 0 \Rightarrow y = \text{const}$. (ii) $\dfrac{dy}{dx} = 3 \Rightarrow y - 3x = \text{const}$.

③ variable transformation. $\begin{cases} \xi = y - 3x \\ \eta = y \end{cases} \Rightarrow$ $u_x = -3u_\xi$, $u_{xy} = -3u_{\xi\xi} - 3u_{\xi\eta}$,
$u_{xx} = -3u_{\xi\xi} \cdot \xi_x = 9u_{\xi\xi}$

④ Canonical form: $yu_{xx} + 3yu_{xy} + 3u_x = 9\eta u_{\xi\xi} + 3\eta(-3u_{\xi\xi} - 3u_{\xi\eta}) - 9u_\xi = 0$,
i.e. $\eta u_{\xi\eta} + u_\xi = 0$.

⑤ general sol^n: Let $u_\xi = v$, then v satisfies $\eta v_\eta + v = 0$ and the sol^n of v is $v = C \cdot \frac{1}{\eta}$, where C is an arbitrary function of ξ. Therefore, $u_\xi = C(\xi) \cdot \frac{1}{\eta}$ and general sol^n of u is $u = \varphi(\xi) \cdot \frac{1}{\eta} + \psi(\eta)$, which means that $u = \frac{1}{y}\varphi(y-3x) + \psi(y)$, where φ and ψ are arbitrary functions.

2. ① The discriminant is: $B^2-4AC = 4x^2y^2 - 4x^2y^2 = 0$.

② characteristic curve: $\dfrac{dy}{dx} = \dfrac{B}{2A} = \dfrac{y}{x} \Rightarrow \dfrac{y}{x} = \text{const}$.

③ variable transformation: $\begin{cases} \xi = \frac{y}{x}, \\ \eta = x. \end{cases} \Rightarrow$ $u_x = \cancel{-\frac{\xi^2}{\eta}u}\ u_\xi \cdot \xi_x + u_\eta \cdot \eta_x$
$= -\frac{y}{x^2}u_\xi + u_\eta = -\frac{\xi}{\eta}u_\xi + u_\eta$

$u_y = u_\xi \cdot \xi_y = \frac{1}{\eta}u_\xi$, $u_{xx} = \left(\frac{\xi}{\eta}\right)^2 u_{\xi\xi} - 2\frac{\xi}{\eta}u_{\xi\eta} + u_{\eta\eta} + \frac{2\xi}{\eta^2}u_\xi$,
$u_{xy} = -\frac{\xi}{\eta^2}u_{\xi\xi} + \frac{1}{\eta}u_{\xi\eta} - \frac{1}{\eta^2}u_\xi$, $u_{yy} = \frac{1}{\eta^2}u_{\xi\xi}$.

④ canonical form: $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \eta^2\left\{\left(\frac{\xi}{\eta}\right)^2 u_{\xi\xi} - 2\frac{\xi}{\eta}u_{\xi\eta} + u_{\eta\eta} + \frac{2\xi}{\eta^2}u_\xi\right\}$
$+ 2\eta^2\xi\left(-\frac{\xi}{\eta^2}u_{\xi\xi} + \frac{1}{\eta}u_{\xi\eta} - \frac{1}{\eta^2}u_\xi\right) + (\xi\eta)^2 \cdot \frac{1}{\eta^2}u_{\xi\xi} = \eta^2 u_{\eta\eta} = 0$

⑤ general sol^n: $u_{\eta\eta} = 0 \Rightarrow u_\eta = C_1(\xi)$ and $u = \eta\varphi(\xi) + \psi(\xi)$, which means that $u = x\varphi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$, where φ and ψ are arbitrary functions.