Consider the problem

$$\begin{cases} u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty,\ y > 0, \\ u_y(x,0) = g(x), \quad -\infty < x < \infty, \\ u \text{ and } u_x \text{ vanish as } |x| \to \infty \text{ and } u(x,y) \text{ is bounded as } y \to \infty. \end{cases}$$

Solution: Applying the Fourier transform with respect to x to the equation, i.e.

$$U(k,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,y) e^{-ikx} dx,$$

one gets

$$(ik)^2 U + U_{yy} = 0, \tag{0.1a}$$

$$U_y(k,0) = G(k), \tag{0.1b}$$

where

$$G(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-ikx} dx.$$

Solving the ordinary differential equation (0.1a), one gets

$$U(k,y) = Ae^{ky} + Be^{-ky},$$

and by differentiating it, one has

$$U_y(k,y) = Ake^{ky} - Bke^{ky} \implies U_y(k,0) = (A-B)k = G(k) \implies (A-B) = \frac{G(k)}{k}.$$

Moreover, since $u(x,y)$ is bounded as $y \to \infty$, $U(k,y)$ is bounded as well as $y \to \infty$. Thus $A = 0$ as $k > 0$ and $B = 0$ as $k < 0$, and

$$U(k,y) = \begin{cases} -\dfrac{G(k)}{k} e^{-ky}, & k > 0, \\ \dfrac{G(k)}{k} e^{ky}, & k < 0. \end{cases}$$

Taking the inverse Fourier transform, one gets

$$\begin{aligned} u(x,y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(k,y) e^{ikx} dk = -\frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{G(k)}{k} e^{-ky+ikx} dk + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{G(k)}{k} e^{ky+ikx} dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(-\int_0^{\infty} \frac{e^{-ky+ik(x-\xi)}}{k} dk + \int_{-\infty}^{0} \frac{e^{ky+ik(x-\xi)}}{k} dk \right) g(\xi) d\xi, \end{aligned}$$

and thus it is to consider the integral with regards to k. Noting that

$$\frac{e^{-ky}}{k} = \int_y^{\infty} e^{-k\eta} d\eta, \quad k > 0$$

$$\frac{e^{ky}}{k} = -\int_y^{\infty} e^{k\eta} d\eta, \quad k < 0$$

one gets

$$\begin{aligned} -\int_0^{\infty} \frac{e^{-ky+ik(x-\xi)}}{k} dk + \int_{-\infty}^{0} \frac{e^{ky+ik(x-\xi)}}{k} dk &= -\int_y^{\infty} \int_0^{\infty} e^{-k\eta+ik(x-\xi)} dk d\eta - \int_y^{\infty} \int_{-\infty}^{0} e^{k\eta+ik(x-\xi)} dk d\eta \\ &= -\int_y^{\infty} \left(\frac{1}{\eta - i(x-\xi)} + \frac{1}{\eta + i(x-\xi)} \right) d\eta = \int_y^{\infty} \frac{2\eta}{\eta^2 + (x-\xi)^2} d\eta, \end{aligned}$$

therefore, the solution is given by

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi)\int_{\infty}^{y} \frac{2\eta}{\eta^2+(x-\xi)^2}d\eta d\xi.$$

Remark 0.1. *Since the integral over η is not well defined, in practice, one should integrate over ξ first and then integrate with respect to η.*